Some properties of (α, β)-interval valued fuzzy ideals in BF-algebras

M. Idrees, A. Rehman, M. Zulfiqar, S. Muhammad

Abstract. **In this paper, we introduce the concept of (α, β)-interval valued fuzzy ideals in BF-algebra, where α, β are any one of $\in$, q, $\in \vee q$, $\in \wedge q$ and investigate some of their related properties. We prove that every $(\in \vee q, \in \vee q)$-interval valued fuzzy ideal of a BF-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X. We show that when an $(\in, \in \vee q)$-interval valued fuzzy ideal of a BF-algebra X is an $(\in, \in)$-interval valued fuzzy ideal of X. We also prove that the intersection and union of any family of $(\in, \in \vee q)$-interval valued fuzzy ideals of a BF-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.**

Key words: BF-algebra; (α, β)-interval valued fuzzy ideals; $(\in, \in \vee q)$-interval valued fuzzy ideal.

I. INTRODUCTION

The concept of BF-algebra was first initiated by Walendziak [25] in 2007. The theory BF-algebra was further enriched by many authors [5, 9, 24].

The fuzzy sets, proposed by Zadeh [22] in 1965, has provided a useful mathematical tool for describing the behavior of systems that are too complex or ill defined to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields, for example, artificial intelligence, computer science, control engineering, expert system, management science, operation research and many others. The concept was applied to the theory of groupoids and groups by Rosenfeld [22], where he introduced the fuzzy subgroup of a group.

A new type of fuzzy subgroup, which is, the $(\in, \in \vee q)$-fuzzy subgroup, was introduced by Bhakat and Das [3] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [21]. Murali [20] proposed the definition of fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subsets. It was found that the most viable generalization of Rosenfeld's fuzzy subgroup is $(\in, \in \vee q)$-fuzzy subgroup. Bhakat [1-2] initiated the concepts of $(\in \vee q)$-level subsets, $(\in, \in \vee q)$-fuzzy normal, quasi-normal and maximal subgroups. Many researchers utilized these concepts to generalize some concepts of algebra (see [4, 7, 8, 26, 31-33]). In [6], Davvaz studied $(\in, \in \vee q)$-fuzzy subnearrings and ideals. In [11-13], Jun defined the notion of (α, β)-fuzzy subalgebras/ideals in BCK/BCI-algebras. The concept of (α, β)-fuzzy positive implicative ideal in BCK-algebras was initiated by Zulfiqar in [31]. In [14], Jun defined $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras. In [32], Zulfiqar introduced the notion of sub-implicative (α, β)-fuzzy ideals in BCH-algebras. Currently, Zulfiqar and Shabir defined the concept of positive implicative $(\in, \in \vee q)$-fuzzy ideals ($(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals, fuzzy ideals with thresholds) in BCK-algebras in [33].

The theory of interval valued fuzzy sets was proposed forty year ago as a natural extension of fuzzy sets. Interval valued fuzzy set was introduced by Zadeh [28], where the value of the membership function is interval of numbers instead of the number. The theory was further enriched by many authors [4, 7-8, 10, 15-19, 23, 29-30]. In [4], Biswas defined interval valued fuzzy subgroups of Rosenfeld's nature, and investigated some elementary properties. Jun, introduced the concept of interval valued fuzzy subalgebras/ideals in BCK-algebras [10]. In [15], Latha et al. initiated the notion of interval valued (α, β)-fuzzy subgroups. In [16], Ma et al. defined the concept of interval valued $(\in, \in \vee q)$-fuzzy ideals of pseudo MV-algebras. In [17-18], Ma et al. studied $(\in, \in \vee q)$-interval valued fuzzy ideals in BCI-algebras. Mostafa et al. initiated the notion of interval valued fuzzy KU-ideals in KU-algebras [19]. In [23], Saeid defined the concept of interval valued fuzzy BG-algebras. Zhan et al. [30] initiated the notion of interval valued $(\in, \in \vee q)$-fuzzy filters of pseudo BL-algebras.

In the present paper, we define the concept of (α, β)-interval valued fuzzy ideals in BF-algebra, where α, β are any one of $\in$, q, $\in \vee q$, $\in \wedge q$ and investigate some of their related properties. We prove that every $(\in \vee q, \in \vee q)$-interval valued fuzzy ideal of a BF-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X. We show that when an $(\in, \in \vee q)$-interval valued fuzzy ideal of a BF-algebra X is an $(\in, \in)$-interval valued fuzzy ideal of X. We also prove that the intersection and union of any family of $(\in, \in \vee q)$-interval valued fuzzy ideals of a BF-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.

2. PRELIMINARIES

Throughout this paper X always denote a BF-algebra without any specification. We also include some basic aspects that are necessary for this paper.

A BF-algebra X [25] is a general algebra (X, $*$, 0) of type (2, 0) satisfying the following conditions:

(BF-1) $x * x = 0$

(BF-2) $x * 0 = x$

(BF-3) $0 * (x * y) = (y * x)$

for all x, y $\in$ X.

We can define a partial order "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$.

Definition 2.1. [25] A nonempty subset S of a BF-algebra X is called a subalgebra of X if it satisfies

$$x * y \in S, \text{ for all } x, y \in S.$$

Definition 2.2. [5] A non-empty subset I of a BF-algebra X is called an ideal of X if it satisfies the conditions (I1) and (I2), where

(I1) $0 \in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$,

for all x, y $\in$ X.

We now review some interval-valued fuzzy logic concepts. First, we denote by $\bar{c} = [c^-, c^+]$ a closed interval of [0, 1], where $0 \leq c^- \leq c^+ \leq 1$ and denote by H[0, 1] the set of all such closed intervals of [0, 1].

Define on H[0, 1] an order relation "$\leq$" by

(1) $\bar{c}_1 \leq \bar{c}_2 \Leftrightarrow c_1^- \leq c_2^-$ and $c_1^+ \leq c_2^+$

(2) $\bar{c}_1 = \bar{c}_2 \Leftrightarrow c_1^- = c_2^-$ and $c_1^+ = c_2^+$

(3) $\bar{c}_1 < \bar{c}_2 \Leftrightarrow \bar{c}_1 \leq \bar{c}_2$ and $\bar{c}_1 \neq \bar{c}_2$

(4) $p\bar{c} = [pc^-, pc^+]$, whenever $0 \leq p \leq 1$

(5) $\text{rmax}\{\bar{c}_i, \bar{d}_i\} = [\max\{c_i^-, d_i^-\}, \max\{c_i^+, d_i^+\}]$

(6) $\text{rmin}\{\bar{c}_i, \bar{d}_i\} = [\min\{c_i^-, d_i^-\}, \min\{c_i^+, d_i^+\}]$

(7) $\text{rinf}\,\bar{c}_i = [\Lambda_{i\in I}\, c_i^-, \Lambda_{i\in I}\, c_i^+]$

(8) $\text{rsup}\,\bar{c}_i = [\vee_{i\in I}\, c_i^-, \vee_{i\in I}\, c_i^+]$

Where

$$\bar{c}_i = [c_i^-, c_i^+],\ \bar{d}_i = [d_i^-, d_i^+] \in H[0, 1],\ i \in I.$$

Then, H[0, 1] with $\leq$ is a complete lattice, with $\wedge$ = rmin, $\vee$ = rmax, $\bar{0} = [0, 0]$ and $\bar{1} = [1, 1]$ being its least element and the greatest element, respectively.

An interval valued fuzzy set $\tilde{\lambda}$ of a universe X is a function from X into the unit closed interval [0, 1], that is $\tilde{\lambda} : X \to H[0, 1]$, $\tilde{\lambda}(x) \in H[0, 1]$, where for each $x \in X$

$$\tilde{\lambda}(x) = [\lambda^-(x), \lambda^+(x)] \in H[0, 1].$$

For an interval valued fuzzy set $\tilde{\lambda}$ in a BF-algebra X and $[0, 0] < \tilde{t} \leq [1, 1]$, the crisp set

$$\lambda_{\tilde{t}} = \{x \in X \mid \tilde{\lambda}(x) \geq \tilde{t}\}$$

is called the level subset of $\tilde{\lambda}$.

We also note that, since every $c \in [0, 1]$ is in correspondence with the interval $[c, c] \in H[0, 1]$, it follows that a fuzzy set is a particular case of interval-valued fuzzy set. First we note that an interval-valued fuzzy set $\tilde{\lambda}$ of a BF-algebra X is a pair of fuzzy sets (λ^-, λ^+) of X such that $\lambda^-(x) \leq \lambda^+(x)$, for all $x \in X$.

If $\bar{C}$, $\bar{D}$ are two interval-valued fuzzy sets of a BF-algebra X, then we define

$\bar{C} \leq \bar{D}$ if and only if for all $x \in X$, $C^-(x) \leq D^-(x)$ and $C^+(x) \leq D^+(x)$,

$\bar{C} = \bar{D}$ if and only if for all $x \in X$, $C^-(x) = D^-(x)$ and $C^+(x) = D^+(x)$.

Also, the union and intersection are defined as follows:

If $\bar{C}$ and $\bar{D}$ are two interval-valued fuzzy sets of a BF-algebra X, where

$\bar{C}(x) = [C^-(x), C^+(x)]$, $\bar{D}(x) = [D^-(x), D^+(x)]$, for all $x \in X$,

then

$(\bar{C} \cup \bar{D})(x) = \bar{C}(x) \vee \bar{D}(x) = [\max\{C^-(x), D^-(x)\}, \max\{C^+(x), D^+(x)\}]$

$(\bar{C} \cap \bar{D})(x) = \bar{C}(x) \wedge \bar{D}(x) = [\max\{C^-(x), D^-(x)\}, \max\{C^+(x), D^+(x)\}]$

$$\overline{C^c}(x) = [1 - C^+(x), 1 - C^-(x)]$$

where the operation "c" is the complement of interval-valued fuzzy set of X.

By the join of two interval-valued fuzzy sets, we know $\bar{C} \cup \overline{C^c}(x) = [\max\{C^-(x), 1 - C^+(x)\}, \max\{C^+(x), 1 - C^-(x)\}]$.

Definition 2.3. An interval valued fuzzy set $\tilde{\lambda}$ of a BF-algebra X is called an interval valued fuzzy ideal of X if it satisfies the conditions (F1) and (F2), where

(F1) $\tilde{\lambda}(0) \geq \tilde{\lambda}(x)$,

(F2) $\tilde{\lambda}(x) \geq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y)$,

for all x, y $\in$ X.

Theorem 2.4. An interval valued fuzzy set $\tilde{\lambda}$ of a BF-algebra X is an interval valued fuzzy ideal of X if and only if, for every $[0, 0] < \tilde{t} \leq [1, 1]$, $\lambda_{\tilde{t}}$ is an ideal of X.

Proof. Assume that $\tilde{\lambda}$ is an interval valued fuzzy ideal of X and let $\tilde{t} \in H[0, 1]$ be such that $x \in \lambda_{\tilde{t}}$. Then

$$\tilde{\lambda}(0) \geq \tilde{\lambda}(x) \geq \tilde{t},$$

we have $0 \in \lambda_{\tilde{t}}$. Let x, y $\in$ X be such that

$$x * y \in \lambda_{\tilde{t}} \text{ and } y \in \lambda_{\tilde{t}}.$$

Then

$$\tilde{\lambda}(x * y) \geq \tilde{t} \text{ and } \tilde{\lambda}(y) \geq \tilde{t}.$$

It follows from F2 that

$$\begin{aligned} \tilde{\lambda}(x) &\geq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y) \\ &\geq \tilde{t} \wedge \tilde{t} \\ &= \tilde{t} \end{aligned}$$

Thus

$$x \in \tilde{\lambda}_{\tilde{t}}.$$

Hence $\tilde{\lambda}_{\tilde{t}}$ is an ideal of X.

Conversely, suppose that $\tilde{\lambda}_{\tilde{t}}$ is an ideal of X for all $[0, 0] < \tilde{t} \leq [1, 1]$.

Assume that there exist $a \in$ X such that

$$\tilde{\lambda}(0) < \tilde{\lambda}(a).$$

Let

$$\tilde{\lambda}(0) = [\lambda^-(0), \lambda^+(0)]$$

and

$$\tilde{\lambda}(a) = [\lambda^-(a), \lambda^+(a)]$$

Then

$$\lambda^-(0) < \lambda^-(a) \text{ and } \lambda^+(0) < \lambda^+(a).$$

If we take

$$\tilde{\delta} = [\delta^-, \delta^+] = \frac{1}{2}(\tilde{\lambda}(0) + \tilde{\lambda}(a)),$$

Then

$$[\delta^-, \delta^+] = [\frac{1}{2}(\lambda^-(0) + \lambda^-(a)), \frac{1}{2}(\lambda^+(0) + \lambda^+(0))].$$

Hence

$$\lambda^-(0) < \lambda^- < \lambda^-(a) \text{ and } \lambda^+(0) < \lambda^+ < \lambda^+(a).$$

This implies that

$$\tilde{\lambda}(0) = [\lambda^-(0), \lambda^+(0)] < [\delta^-, \delta^+] < [\lambda^-(a), \lambda^+(a)].$$

This shows that $0 \notin \tilde{\lambda}_{\tilde{\delta}}$, which is contradiction. Therefore

$$\tilde{\lambda}(0) \geq \tilde{\lambda}(x)$$

for all $x \in$ X. Now, let us suppose there are a, $b \in$ X such that

$$\tilde{\lambda}(a) < \tilde{\lambda}(a * b) \wedge \tilde{\lambda}(b).$$

Let

$$\tilde{\lambda}(a) = [\lambda^-(a), \lambda^+(a)],$$

$$\tilde{\lambda}(a * b) = [\lambda^-(a * b), \lambda^+(a * b)]$$

and

$$\tilde{\lambda}(b) = [\lambda^-(b), \lambda^+(b)]$$

Put

$$\tilde{\gamma} = [\gamma^-, \gamma^+] = \frac{1}{2}(\tilde{\lambda}(a) + (\tilde{\lambda}(a * b) \wedge \tilde{\lambda}(b)))$$

Then

$$\lambda^-(a) < \gamma^- < \lambda^-(a * b) \wedge \lambda^-(b)$$

and

$$\lambda^+(a) < \gamma^+ < \lambda^+(a * b) \wedge \lambda^+(b)$$

Then we will have

$$\tilde{\lambda}(a) = [\lambda^-(a), \lambda^+(a)] < [\gamma^-, \gamma^+] <$$
$$[\lambda^-(a * b) \wedge \lambda^-(b), \lambda^+(a * b) \wedge \lambda^+(b)].$$

Therefore

$$a \notin \tilde{\lambda}_{\tilde{\gamma}}.$$

But

$$\tilde{\lambda}(a * b) = [\lambda^-(a * b), \lambda^+(a * b)] > \tilde{\gamma}$$

and

$$\tilde{\lambda}(b) = [\lambda^-(b), \lambda^+(b)] > \tilde{\gamma}$$

which implies that

$$a * b \in \tilde{\lambda}_{\tilde{\gamma}} \quad \text{and} \quad b \in \tilde{\lambda}_{\tilde{\gamma}}$$

This leads to a contradiction.

Hence

$$\tilde{\lambda}(x) \geq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y).$$

Thus $\tilde{\lambda}$ is an interval valued fuzzy ideal of X.

3. (α, β)-INTERVAL VALUED FUZZY IDEALS IN BF-ALGEBRA

In this section, we define the concept of (α, β)-interval valued fuzzy ideals in a BF-algebra and investigate some of their properties. Throughout this paper X will denote a BF-algebra and α, β are any one of $\in$, q, $\in \vee q$, $\in \wedge q$ unless otherwise specified.

An interval valued fuzzy set $\tilde{\lambda}$ of a BF-algebra X having the form

$$\tilde{\lambda}(y) = \begin{cases} \tilde{t} \ (\neq [0, 0]) & \text{if } y = x \\ [0, 0] & \text{if } y \neq x \end{cases}$$

is said to be an interval valued fuzzy point with support x and value $\tilde{t}$ and is denoted by $x_{\tilde{t}}$. An interval valued fuzzy point $x_{\tilde{t}}$ is said to belong to (resp., quasi-coincident with) an interval valued fuzzy set $\tilde{\lambda}$, written as $x_{\tilde{t}} \in \tilde{\lambda}$ (resp. $x_{\tilde{t}}\, q\, \tilde{\lambda}$) if $\tilde{\lambda}(x) \geq \tilde{t}$ (resp. $\tilde{\lambda}(x) + \tilde{t} > [1, 1]$). By $x_{\tilde{t}} \in \vee q\, \tilde{\lambda}$ ($x_{\tilde{t}} \in \wedge q\, \tilde{\lambda}$) we mean that $x_{\tilde{t}} \in \tilde{\lambda}$ or $x_{\tilde{t}}\, q\, \tilde{\lambda}$ ($x_{\tilde{t}} \in \tilde{\lambda}$ and $x_{\tilde{t}}\, q\, \tilde{\lambda}$).

In what follows let α and β denote any one of $\in$, q, $\in \vee q$, $\in \wedge q$ and $\alpha \neq \in \wedge q$ unless otherwise specified. To say that $x_{\tilde{t}}\, \overline{\alpha}\, \tilde{\lambda}$ means that $x_{\tilde{t}}\, \alpha\, \tilde{\lambda}$ does not hold.

Definition 3.1. An interval valued fuzzy set $\tilde{\lambda}$ of a BF-algebra X is called an (α, β)-fuzzy subalgebra of X, where $\alpha \neq \in \wedge q$, if it satisfies the condition

$$x_{\tilde{t}_1}\, \alpha\, \tilde{\lambda},\ y_{\tilde{t}_2}\, \alpha\, \tilde{\lambda} \ \Rightarrow\ (x * y)_{\tilde{t}_1 \wedge \tilde{t}_2}\, \beta\, \tilde{\lambda}$$

for all $[0, 0] < \tilde{t}_1, \tilde{t}_2 \leq [1, 1]$ and $x, y \in$ X.

Let $\tilde{\lambda}$ be an interval valued fuzzy set of a BF-algebra X such that $\tilde{\lambda}(x) \leq [0.5, 0.5]$ for all $x \in$ X. Let $x \in$ X and $[0, 0] < \tilde{t} \leq [1, 1]$ be such that

$$x_{\tilde{t}} \in \wedge q\, \tilde{\lambda}.$$

Then

$$\tilde{\lambda}(x) \geq \tilde{t} \text{ and } \tilde{\lambda}(x) + \tilde{t} > [1, 1].$$

It follows that

$$2\tilde{\lambda}(x) = \tilde{\lambda}(x) + \tilde{\lambda}(x) \geq \tilde{\lambda}(x) + \tilde{t} > [1, 1].$$

This implies that $\tilde{\lambda}$ (x) > [0.5, 0.5]. This means that

$$\{ x_{\tilde{t}} \mid x_{\tilde{t}} \in \wedge q \tilde{\lambda} \} = \phi .$$

Therefore, the case $\alpha = \in \wedge q$ in the above definition is omitted.

Definition 3.2. An interval valued fuzzy set $\tilde{\lambda}$ of a BF-algebra X is called an (α, β)-interval valued fuzzy ideal of X, where $\alpha \neq \in \wedge q$, if it satisfies the conditions (A) and (B), where

(A) $x_{\tilde{t}} \ \alpha \ \tilde{\lambda} \ \Rightarrow \ 0_{\tilde{t}} \ \beta \ \tilde{\lambda}$,

(B) $(x * y)_{\tilde{t}_1} \ \alpha \ \tilde{\lambda}, \ y_{\tilde{t}_2} \ \alpha \ \tilde{\lambda} \ \Rightarrow \ x_{\tilde{t}_1 \wedge \tilde{t}_2} \ \beta \ \tilde{\lambda}$,

for all $[0, 0] < \tilde{t}, \tilde{t}_1, \tilde{t}_2 \leq [1, 1]$ and x, y $\in$ X.

Theorem 3.3. For any interval valued fuzzy set $\tilde{\lambda}$ of a BF-algebra X, the condition (F1) and (F3) are equivalent to the conditions:

(C) $x_{\tilde{t}} \in \tilde{\lambda} \ \Rightarrow \ 0_{\tilde{t}} \in \tilde{\lambda}$,

(D) $(x * y)_{\tilde{t}_1} \in \tilde{\lambda}, \ y_{\tilde{t}_2} \in \tilde{\lambda} \ \Rightarrow \ x_{\tilde{t}_1 \wedge \tilde{t}_2} \in \tilde{\lambda}$,

for all $[0, 0] < \tilde{t}, \tilde{t}_1, \tilde{t}_2 \leq [1, 1]$ and x, y, z $\in$ X.

Proof. (F1) $\Rightarrow$ (C)

Let x $\in$ X and $[0, 0] < \tilde{t} \leq [1, 1]$ be such that $x_{\tilde{t}} \in \tilde{\lambda}$, that is

$$\tilde{\lambda}(x) \geq \tilde{t} .$$

Then by (F1)

$$\tilde{\lambda}(0) \geq \tilde{\lambda}(x) \geq \tilde{t}$$

and so $0_{\tilde{t}} \in \tilde{\lambda}$.

(C) $\Rightarrow$ (F1)

Since $x_{\tilde{\lambda}(x)} \in \tilde{\lambda}$, for x $\in$ X. Thus by hypothesis $0_{\tilde{\lambda}(x)} \in \tilde{\lambda}$, so we have

$$\tilde{\lambda}(0) \geq \tilde{\lambda}(x).$$

(F2) $\Rightarrow$ (D)

Let x, y $\in$ X and $[0, 0] < \tilde{t}_1, \tilde{t}_2 \leq [1, 1]$ be such that

$$(x * y)_{\tilde{t}_1} \in \tilde{\lambda} \text{ and } y_{\tilde{t}_2} \in \tilde{\lambda} .$$

Then

$$\tilde{\lambda}(x * y) \geq \tilde{t}_1 \text{ and } \tilde{\lambda}(y) \geq \tilde{t}_2 .$$

By (F2)

$$\tilde{\lambda}(x) \geq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y) \geq \tilde{t}_1 \wedge \tilde{t}_2 .$$

Thus

$$x_{\tilde{t}_1 \wedge \tilde{t}_2} \in \tilde{\lambda} .$$

(D) $\Rightarrow$ (F2)

Note that for every x, y $\in$ X, we have

$$(x * y)_{\tilde{\lambda}(x * y)} \in \tilde{\lambda} \text{ and } y_{\tilde{\lambda}(y)} \in \tilde{\lambda} .$$

Hence by hypothesis

$$x_{\tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y)} \in \tilde{\lambda} .$$

This implies that

$$\tilde{\lambda}(x) \geq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y).$$

Theorem 3.4. Every $(\in \vee q, \in \vee q)$-interval valued fuzzy ideal of a BF-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.

Proof. Let $\tilde{\lambda}$ be an $(\in \vee q, \in \vee q)$-interval valued fuzzy ideal of X. Let x $\in$ X and $[0, 0] < \tilde{t} \leq [1, 1]$ be such that $x_{\tilde{t}} \in \tilde{\lambda}$. Then

$$x_{\tilde{t}} \in \vee q \tilde{\lambda}$$

and so

$$0_{\tilde{t}} \in \vee q \tilde{\lambda} .$$

Let x, y $\in$ X and $[0, 0] < \tilde{t}_1, \tilde{t}_2 \leq [1, 1]$ be such that

$$(x * y)_{\tilde{t}_1} \in \tilde{\lambda} \text{ and } y_{\tilde{t}_2} \in \tilde{\lambda} .$$

Then

$$(x * y)_{\tilde{t}_1} \in \vee q \tilde{\lambda} \text{ and } y_{\tilde{t}_2} \in \vee q \tilde{\lambda} .$$

This implies that

$$x_{\tilde{t}_1 \wedge \tilde{t}_2} \in \vee q \tilde{\lambda} .$$

Therefore $\tilde{\lambda}$ is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.

Theorem 3.5. An interval valued fuzzy set $\tilde{\lambda}$ of a BF-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X if and only if it satisfies the conditions (E) and (F), where

(E) $\tilde{\lambda}(0) \geq \tilde{\lambda}(x) \wedge [0.5, 0.5]$,

(F) $\tilde{\lambda}(x) \geq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5]$,

for all x, y $\in$ X.

Proof. Suppose $\tilde{\lambda}$ is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X. Let x $\in$ X be such that

$$\tilde{\lambda}(0) < \tilde{\lambda}(x) \wedge [0.5, 0.5].$$

If $\tilde{\lambda}$ (x) < [0.5, 0.5], then

$$\tilde{\lambda}(0) < \tilde{\lambda}(x).$$

Select $[0, 0] < \tilde{t} \leq [0.5, 0.5]$ such that

$$\tilde{\lambda}(0) < \tilde{t} \leq \tilde{\lambda}(x).$$

Then

$$x_{\tilde{t}} \in \tilde{\lambda} \text{ but } 0_{\tilde{t}} \ \overline{\in \vee q} \ \tilde{\lambda},$$

which is a contradiction.

If $\tilde{\lambda}$ (x) $\geq$ [0.5, 0.5], then $\tilde{\lambda}$ (0) < [0.5, 0.5]. This implies that

$$x_{[0.5,\, 0.5]} \in \tilde{\lambda} \text{ but } 0_{[0.5,\, 0.5]} \ \overline{\in \vee q} \ \tilde{\lambda} .$$

Again a contradiction. Hence

$$\tilde{\lambda}(0) \geq \tilde{\lambda}(x) \wedge [0.5, 0.5],$$

for all x $\in$ X.

Now we show that condition (F) holds. On the contrary assume that there exist x, y $\in$ X such that

$$\tilde{\lambda}(x) < \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5].$$

If $\tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y) < [0.5, 0.5]$, then

$$\tilde{\lambda}(x) < \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y).$$

Select $[0, 0] < \tilde{t} \leq [0.5, 0.5]$ such that

$$\tilde{\lambda}(x) < \tilde{t} \leq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y).$$

Then

$$(x * y)_{\tilde{t}} \in \tilde{\lambda} \text{ and } y_{\tilde{t}} \in \tilde{\lambda} \text{ but } x_{\tilde{t}} \,\overline{\in \vee\, q}\, \tilde{\lambda},$$

which is a contradiction.

If $\tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y) \geq [0.5, 0.5]$, then

$$\tilde{\lambda}(x) < [0.5, 0.5].$$

This implies

$$(x * y)_{[0.5, 0.5]} \in \tilde{\lambda} \text{ and } y_{[0.5, 0.5]} \in \tilde{\lambda} \text{ but } x_{[0.5, 0.5]} \,\overline{\in \vee\, q}\, \tilde{\lambda}.$$

Again a contradiction. Hence

$$\tilde{\lambda}(x) \geq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5].$$

Conversely, assume that $\tilde{\lambda}$ satisfies the conditions (E) and (F). Let x $\in$ X and $[0, 0] < \tilde{t} \leq [1, 1]$ be such that $x_{\tilde{t}} \in \tilde{\lambda}$. Then $\tilde{\lambda}(x) \geq \tilde{t}$. By condition (E), we have

$$\begin{aligned}\tilde{\lambda}(0) &\geq \tilde{\lambda}(x) \wedge [0.5, 0.5]\\ &\geq \tilde{t} \wedge [0.5, 0.5].\end{aligned}$$

If $\tilde{t} \leq [0.5, 0.5]$, then $\tilde{\lambda}(0) \geq \tilde{t}$. This implies $0_{\tilde{t}} \in \tilde{\lambda}$.

If $\tilde{t} > [0.5, 0.5]$, then $\tilde{\lambda}(0) \geq [0.5, 0.5]$. This implies

$$\tilde{\lambda}(0) + \tilde{t} > [0.5, 0.5] + [0.5, 0.5] = [1, 1],$$

that is, $0_{\tilde{t}}\, q\, \tilde{\lambda}$. Hence

$$0_{\tilde{t}} \in \vee\, q\, \tilde{\lambda}.$$

Let x, y $\in$ X and $[0, 0] < \tilde{t}_1, \tilde{t}_2 \leq [1, 1]$ be such that

$$(x * y)_{\tilde{t}_1} \in \tilde{\lambda} \text{ and } y_{\tilde{t}_2} \in \tilde{\lambda}.$$

Then

$$\tilde{\lambda}(x * y) \geq \tilde{t}_1 \text{ and } \tilde{\lambda}(y) \geq \tilde{t}_2.$$

By condition (F), we have

$$\begin{aligned}\tilde{\lambda}(x) &\geq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5]\\ &\geq \tilde{t}_1 \wedge \tilde{t}_2 \wedge [0.5, 0.5].\end{aligned}$$

If $\tilde{t}_1 \wedge \tilde{t}_2 \leq [0.5, 0.5]$, then

$$\tilde{\lambda}(x) \geq \tilde{t}_1 \wedge \tilde{t}_2.$$

This implies

$$x_{\tilde{t}_1 \wedge \tilde{t}_2} \in \tilde{\lambda}.$$

If $\tilde{t}_1 \wedge \tilde{t}_2 > [0.5, 0.5]$, then

$$\tilde{\lambda}(x) \geq [0.5, 0.5].$$

This implies

$$\tilde{\lambda}(x) + \tilde{t}_1 \wedge \tilde{t}_2 > [0.5, 0.5] + [0.5, 0.5] = [1, 1],$$

i.e.,

$$x_{\tilde{t}_1 \wedge \tilde{t}_2}\, q\, \tilde{\lambda}.$$

Hence

$$x_{\tilde{t}_1 \wedge \tilde{t}_2} \in \vee\, q\, \tilde{\lambda}.$$

This shows that $\tilde{\lambda}$ is an $(\in, \in \vee\, q)$-interval valued fuzzy ideal of X.

Theorem 3.6. An interval valued fuzzy set $\tilde{\lambda}$ of a BF-algebra X is an interval valued fuzzy ideal of X if and only if $\tilde{\lambda}$ is an $(\in, \in)$-interval valued fuzzy ideal of X.

Proof. Suppose $\tilde{\lambda}$ is an interval valued fuzzy ideal of X and $x_{\tilde{t}} \in \tilde{\lambda}$ for x $\in$ X and $[0, 0] < \tilde{t} \leq [1, 1]$. Then $\tilde{\lambda}(x) \geq \tilde{t}$. By Definition 2.3, $\tilde{\lambda}(0) \geq \tilde{\lambda}(x)$, we have

$$\tilde{\lambda}(0) \geq \tilde{t},$$

that is

$$0_{\tilde{t}} \in \tilde{\lambda}.$$

Let x, y $\in$ X and $[0, 0] < \tilde{t}, \tilde{r} \leq [1, 1]$ be such that

$$(x * y)_{\tilde{t}} \in \tilde{\lambda} \text{ and } y_{\tilde{r}} \in \tilde{\lambda}.$$

Then

$$\tilde{\lambda}(x * y) \geq \tilde{t} \text{ and } \tilde{\lambda}(y) \geq \tilde{r}.$$

By Definition 2.3, we have

$$\begin{aligned}\tilde{\lambda}(x) &\geq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y)\\ &\geq \tilde{t} \wedge \tilde{r}.\end{aligned}$$

This implies that

$$x_{\tilde{t} \wedge \tilde{r}} \in \tilde{\lambda}.$$

This shows that $\tilde{\lambda}$ is an $(\in, \in)$-interval valued fuzzy ideal of X.

Conversely, assume that $\tilde{\lambda}$ is an $(\in, \in)$-interval valued fuzzy ideal of X. Suppose there exists x $\in$ X such that

$$\tilde{\lambda}(0) < \tilde{\lambda}(x).$$

Select $[0, 0] < \tilde{t} \leq [1, 1]$ such that

$$\tilde{\lambda}(0) < \tilde{t} \leq \tilde{\lambda}(x).$$

Then $x_{\tilde{t}} \in \tilde{\lambda}$ but $0_{\tilde{t}} \,\overline{\in}\, \tilde{\lambda}$, which is a contradiction. Hence

$$\tilde{\lambda}(0) \geq \tilde{\lambda}(x), \text{ for all } x \in X.$$

Now suppose there exist x, y $\in$ X such that

$$\tilde{\lambda}(x) < \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y).$$

Select $[0, 0] < \tilde{t} \leq [1, 1]$ such that

$$\tilde{\lambda}(x) < \tilde{t} \leq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y).$$

Then $(x * y)_{\tilde{t}} \in \tilde{\lambda}$ and $y_{\tilde{t}} \in \tilde{\lambda}$ but $x_{\tilde{t}} \,\overline{\in}\, \tilde{\lambda}$, which is a contradiction. Hence

$$\tilde{\lambda}(x) \geq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y).$$

This shows that $\tilde{\lambda}$ is an interval valued fuzzy ideal of X.

Theorem 3.7. Every $(\in, \in)$-interval valued fuzzy ideal of a BF-algebra X is an $(\in, \in \vee\, q)$-interval valued fuzzy ideal of X.

Proof. Straightforward.

Corollary 3.8. Every interval valued fuzzy ideal of a BF-algebra X is an $(\in, \in \vee\, q)$-interval valued fuzzy ideal of X.

Proof. By Theorem 3.6, every interval valued fuzzy ideal of a BF-algebra X is an $(\in, \in)$-interval valued fuzzy ideal of X. Hence by above Theorem 3.7, every interval valued fuzzy ideal of X is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.

Next we show that when an $(\in, \in \vee q)$-interval valued fuzzy ideal of a BF-algebra X is an $(\in, \in)$-interval valued fuzzy ideal of X.

Theorem 3.9. Let $\tilde{\lambda}$ be an $(\in, \in \vee q)$-interval valued fuzzy ideal of a BF-algebra X such that $\tilde{\lambda}(x) < [0.5, 0.5]$ for all $x \in X$. Then $\tilde{\lambda}$ is an $(\in, \in)$-interval valued fuzzy ideal of X.

Proof. Let $x \in X$ and $[0, 0] < \tilde{t} \leq [1, 1]$ be such that

$$x_{\tilde{t}} \in \tilde{\lambda}.$$

Then

$$\tilde{\lambda}(x) \geq \tilde{t}.$$

Since

$$\begin{aligned}\tilde{\lambda}(0) &\geq \tilde{\lambda}(x) \wedge [0.5, 0.5] \\ &= \tilde{\lambda}(x) \\ &\geq \tilde{t}\end{aligned}$$

we have

$$0_{\tilde{t}} \in \tilde{\lambda}.$$

Now let $x, y \in X$ and $[0, 0] < \tilde{t}_1, \tilde{t}_2 \leq [1, 1]$ be such that

$$(x * y)_{\tilde{t}_1} \in \tilde{\lambda} \text{ and } y_{\tilde{t}_2} \in \tilde{\lambda}.$$

Then

$$\tilde{\lambda}(x * y) \geq \tilde{t}_1 \text{ and } \tilde{\lambda}(y) \geq \tilde{t}_2.$$

It follows from Theorem 3.5(F) that

$$\begin{aligned}\tilde{\lambda}(x) &\geq \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y) \wedge [0.5, 0.5] \\ &= \tilde{\lambda}(x * y) \wedge \tilde{\lambda}(y) \\ &\geq \tilde{t}_1 \wedge \tilde{t}_2.\end{aligned}$$

Thus

$$x_{\tilde{t}_1 \wedge \tilde{t}_2} \in \tilde{\lambda}.$$

Therefore $\tilde{\lambda}$ is an $(\in, \in)$-interval valued fuzzy ideal of X.

Corollary 3.10. Let $\tilde{\lambda}$ be an $(\in, \in \vee q)$-interval valued fuzzy ideal of a BF-algebra X such that $\tilde{\lambda}(x) < [0.5, 0.5]$ for all $x \in X$. Then $\tilde{\lambda}$ is an interval valued fuzzy ideal of X.

Theorem 3.11. Let I be an ideal of a BF-algebra X. Then the interval valued fuzzy set $\tilde{\lambda}$ of X defined by

$$\tilde{\lambda}(x) = \begin{cases} \geq [0.5, 0.5] & \text{if } x \in I \\ [0, 0] & \text{otherwise,} \end{cases}$$

is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.

Proof. Let I be an ideal of X. Let $x \in X$ and $[0, 0] < \tilde{t} \leq [1, 1]$ be such that $x_{\tilde{t}} \in \tilde{\lambda}$. Then

$$\tilde{\lambda}(x) \geq \tilde{t} > [0, 0].$$

Thus

$$\tilde{\lambda}(x) \geq [0.5, 0.5].$$

This implies that $x \in I$. Since I is an ideal of X. So $0 \in I$. Hence

$$\tilde{\lambda}(0) \geq [0.5, 0.5].$$

If $\tilde{t} \leq [0.5, 0.5]$, then

$$\tilde{\lambda}(0) \geq [0.5, 0.5] \geq \tilde{t}.$$

This implies that $0_{\tilde{t}} \in \tilde{\lambda}$.

If $\tilde{t} > [0.5, 0.5]$, then

$$\tilde{\lambda}(0) + \tilde{t} \geq [0.5, 0.5] + [0.5, 0.5] > [1, 1]$$

and so $0_{\tilde{t}}\, q\, \tilde{\lambda}$. This implies that

$$0_{\tilde{t}} \in \vee q\, \tilde{\lambda}.$$

Let $x, y \in X$ and $[0, 0] < \tilde{t}, \tilde{r} \leq [1, 1]$ be such that

$$(x * y)_{\tilde{t}} \in \tilde{\lambda} \text{ and } y_{\tilde{r}} \in \tilde{\lambda}.$$

Then

$$\tilde{\lambda}(x * y) \geq \tilde{t} > [0, 0] \text{ and } \tilde{\lambda}(y) \geq \tilde{r} > [0, 0].$$

Thus

$$\tilde{\lambda}(x * y) \geq [0.5, 0.5] \text{ and } \tilde{\lambda}(y) \geq [0.5, 0.5].$$

This implies that

$$(x * y) \in I \text{ and } y \in I.$$

Since I is an ideal of X. So

$$x \in I.$$

Hence

$$\tilde{\lambda}(x) \geq [0.5, 0.5].$$

If $\tilde{t} \wedge \tilde{r} \leq [0.5, 0.5]$, then

$$\tilde{\lambda}(x) \geq [0.5, 0.5] \geq \tilde{t} \wedge \tilde{r}.$$

This implies that

$$x_{\tilde{t} \wedge \tilde{r}} \in \tilde{\lambda}.$$

If $\tilde{t} \wedge \tilde{r} > [0.5, 0.5]$, then

$$\tilde{\lambda}(x) + \tilde{t} \wedge \tilde{r} \geq [0.5, 0.5] + [0.5, 0.5] > [1, 1]$$

and so

$$x_{\tilde{t} \wedge \tilde{r}}\, q\, \tilde{\lambda}.$$

Thus

$$x_{\tilde{t} \wedge \tilde{r}} \in \vee q\, \tilde{\lambda}.$$

Hence $\tilde{\lambda}$ is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.

Theorem 3.12. Let I be a non-empty subset of a BF-algebra X. Then I is an ideal of X if the interval valued fuzzy set $\tilde{\lambda}$ of X defined by

$$\tilde{\lambda}(x) = \begin{cases} \geq [0.5, 0.5] & \text{if } x \in I \\ [0, 0] & \text{otherwise,} \end{cases}$$

is an $(q, \in \vee q)$-interval valued fuzzy ideal of X.

Proof. Let I be an ideal of X. Let x $\in$ X and $[0, 0] < \tilde{t} \leq [1, 1]$ be such that

$$x_{\tilde{t}}\, q\, \tilde{\lambda}.$$

Then

$$\tilde{\lambda}(x) + \tilde{t} > [1, 1]$$

So x $\in$ I. Since I is an ideal of X. So 0 $\in$ I. Hence

$$\tilde{\lambda}(0) \geq [0.5, 0.5].$$

If $\tilde{t} \leq [0.5, 0.5]$, then

$$\tilde{\lambda}(0) \geq [0.5, 0.5] \geq \tilde{t}.$$

This implies that $0_{\tilde{t}} \in \lambda$.

If $\tilde{t} > [0.5, 0.5]$, then

$$\tilde{\lambda}(0) + \tilde{t} \geq [0.5, 0.5] + [0.5, 0.5] > [1, 1]$$

and so $0_{\tilde{t}}\, q\, \tilde{\lambda}$. This implies that

$$0_{\tilde{t}} \in \vee q\, \tilde{\lambda}.$$

Let x, y $\in$ X and $[0, 0] < \tilde{t}, \tilde{r} \leq [1, 1]$ be such that

$$(x * y)_{\tilde{t}}\, q\, \tilde{\lambda} \text{ and } y_{\tilde{r}}\, q\, \tilde{\lambda}.$$

Then

$$\tilde{\lambda}(x * y) + \tilde{t} > [1, 1] \text{ and } \tilde{\lambda}(y) + \tilde{r} > [1, 1].$$

So

$$(x * y) \in I \text{ and } y \in I.$$

Since I is an ideal of X. So

$$x \in I.$$

Thus

$$\tilde{\lambda}(x) \geq [0.5, 0.5].$$

If $\tilde{t} \wedge \tilde{r} \leq [0.5, 0.5]$, then

$$\tilde{\lambda}(x) \geq [0.5, 0.5] \geq \tilde{t} \wedge \tilde{r}.$$

So

$$x_{\tilde{t} \wedge \tilde{r}} \in \tilde{\lambda}.$$

If $\tilde{t} \wedge \tilde{r} > 0.5$, then

$$\tilde{\lambda}(x) + \tilde{t} \wedge \tilde{r} \geq [0.5, 0.5] + [0.5, 0.5] > [1, 1]$$

and so

$$x_{\tilde{t} \wedge \tilde{r}}\, q\, \tilde{\lambda}.$$

Thus

$$x_{\tilde{t} \wedge \tilde{r}} \in \vee q\, \tilde{\lambda}.$$

Hence $\tilde{\lambda}$ is an $(q, \in \vee q)$-interval valued fuzzy ideal of X.

Theorem 3.13. Let I be a non-empty subset of a BF-algebra X. Then I is an ideal of X if the interval valued fuzzy set $\tilde{\lambda}$ of X defined by

$$\tilde{\lambda}(x) = \begin{cases} \geq [0.5, 0.5] & if\ \ x \in I \\ [0, 0] & otherwise, \end{cases}$$

is an $(\in \vee q, \in \vee q)$-interval valued fuzzy ideal of X.

Proof. Let I be an ideal of X. Let x $\in$ X and $[0, 0] < \tilde{t} \leq [1, 1]$ be such that

$$x_{\tilde{t}} \in \vee q\, \tilde{\lambda}.$$

This implies that

$$x_{\tilde{t}} \in \tilde{\lambda} \text{ or } x_{\tilde{t}}\, q\, \tilde{\lambda}.$$

If $x_{\tilde{t}}\, q\, \tilde{\lambda}$. This implies that

$$\tilde{\lambda}(x) + \tilde{t} > [1, 1].$$

This implies that x $\in$ I. Since I is an ideal of X. So 0 $\in$ I. Hence

$$\tilde{\lambda}(0) \geq [0.5, 0.5].$$

If $\tilde{t} \leq [0.5, 0.5]$, then

$$\tilde{\lambda}(0) \geq [0.5, 0.5] \geq \tilde{t}.$$

This implies that $0_{\tilde{t}} \in \tilde{\lambda}$.

If $\tilde{t} > [0.5, 0.5]$, then

$$\tilde{\lambda}(0) + \tilde{t} \geq [0.5, 0.5] + [0.5, 0.5] > [1, 1]$$

and so $0_{\tilde{t}}\, q\, \tilde{\lambda}$. This implies that

$$0_{\tilde{t}} \in \vee q\, \tilde{\lambda}.$$

Let x, y $\in$ X and $[0, 0] < \tilde{t}, \tilde{r} \leq [1, 1]$ be such that

$$(x * y)_{\tilde{t}} \in \vee q\, \tilde{\lambda} \text{ and } y_{\tilde{r}} \in \vee q\, \tilde{\lambda}.$$

This implies that

$$(x * y)_{\tilde{t}} \in \tilde{\lambda} \text{ or } (x * y)_{\tilde{t}}\, q\, \tilde{\lambda}$$

and

$$y_{\tilde{r}} \in \tilde{\lambda} \text{ or } y_{\tilde{r}}\, q\, \tilde{\lambda}.$$

If $(x * y)_{\tilde{t}}\, q\, \tilde{\lambda}$ and $y_{\tilde{r}}\, q\, \tilde{\lambda}$. This implies that

$$\tilde{\lambda}(x * y) + \tilde{t} > [1, 1] \text{ and } \tilde{\lambda}(y) + \tilde{r} > [1, 1].$$

So

$$(x * y) \in I \text{ and } y \in I.$$

Since I is an ideal of X. So

$$x \in I.$$

Thus

$$\tilde{\lambda}(x) \geq [0.5, 0.5].$$

If $\tilde{t} \wedge \tilde{r} \leq [0.5, 0.5]$, then

$$\tilde{\lambda}(x) \geq [0.5, 0.5] \geq \tilde{t} \wedge \tilde{r}.$$

So

$$x_{\tilde{t} \wedge \tilde{r}} \in \tilde{\lambda}.$$

If $\tilde{t} \wedge \tilde{r} > [0.5, 0.5]$, then

$$\tilde{\lambda}(x) + \tilde{t} \wedge \tilde{r} \geq [0.5, 0.5] + [0.5, 0.5] > [1, 1]$$

and so

$$x_{\tilde{t} \wedge \tilde{r}}\, q\, \tilde{\lambda}.$$

Thus

$$x_{\tilde{t} \wedge \tilde{r}} \in \vee q\, \tilde{\lambda}.$$

Hence $\tilde{\lambda}$ is an $(\in \vee q, \in \vee q)$-interval valued fuzzy ideal of X.

Theorem 3.14. The intersection of any family of $(\in, \in \vee q)$-interval valued fuzzy ideals of a BF-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.

Proof. Let $\{\tilde{\lambda}_i\}_{i \in I}$ be a family of $(\in, \in \vee q)$-interval valued fuzzy ideals of a BF-algebra X and x $\in$ X. So

$$\tilde{\lambda}_i(0) \geq \tilde{\lambda}_i(x) \wedge [0.5, 0.5]$$

for all i $\in$ I. Thus

$$(\wedge_{i \in I} \tilde{\lambda}_i)(0) = \wedge_{i \in I} (\tilde{\lambda}_i(0))$$

$$\geq \wedge_{i \in I} (\tilde{\lambda}_i(x) \wedge [0.5, 0.5])$$

$$= (\wedge_{i \in I} \tilde{\lambda}_i)(x) \wedge [0.5, 0.5].$$

Thus

$$(\wedge_{i \in I} \tilde{\lambda}_i)(0) \geq (\wedge_{i \in I} \tilde{\lambda}_i)(x) \wedge [0.5, 0.5].$$

Let x, y $\in$ X. Since each $\tilde{\lambda}_i$ is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X. So

$$\tilde{\lambda}_i(x) \geq \tilde{\lambda}_i(x * y) \wedge \tilde{\lambda}_i(y) \wedge [0.5, 0.5]$$

for all i $\in$ I. Thus

$$(\wedge_{i \in I} \tilde{\lambda}_i)(x) = \wedge_{i \in I} (\tilde{\lambda}_i(x))$$

$$\geq \wedge_{i \in I} (\tilde{\lambda}_i(x * y) \wedge \tilde{\lambda}_i(y) \wedge [0.5, 0.5])$$

$$= (\wedge_{i \in I} \tilde{\lambda}_i)(x * y) \wedge (\wedge_{i \in I} \tilde{\lambda}_i)(y) \wedge [0.5, 0.5].$$

Therefore

$$(\wedge_{i \in I} \tilde{\lambda}_i)(x) \geq (\wedge_{i \in I} \tilde{\lambda}_i)(x * y) \wedge (\wedge_{i \in I} \tilde{\lambda}_i)(y) \wedge [0.5, 0.5].$$

Hence, $\wedge_{i \in I} \tilde{\lambda}_i$ is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.

Theorem 3.15. The union of any family of $(\in, \in \vee q)$-interval valued fuzzy ideals of a BF-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.

Proof. Let $\{\tilde{\lambda}_i\}_{i \in I}$ be a family of $(\in, \in \vee q)$-interval valued fuzzy ideals of a BF-algebra X and x $\in$ X. So

$$\tilde{\lambda}_i(0) \geq \tilde{\lambda}_i(x) \wedge [0.5, 0.5]$$

for all i $\in$ I. Thus

$$(\vee_{i \in I} \tilde{\lambda}_i)(0) = \vee_{i \in I} (\tilde{\lambda}_i(0))$$

$$\geq \vee_{i \in I} (\tilde{\lambda}_i(x) \wedge [0.5, 0.5])$$

$$= (\vee_{i \in I} \tilde{\lambda}_i)(x) \wedge [0.5, 0.5].$$

Thus

$$(\vee_{i \in I} \tilde{\lambda}_i)(0) \geq (\vee_{i \in I} \tilde{\lambda}_i)(x) \wedge [0.5, 0.5].$$

Let x, y $\in$ X. Since each $\tilde{\lambda}_i$ is an $(\in, \in \vee q)$-interval valued fuzzy p-ideal of X. So

$$\tilde{\lambda}_i(x) \geq \tilde{\lambda}_i(x * y) \wedge \tilde{\lambda}_i(y) \wedge [0.5, 0.5]$$

for all i $\in$ I. Thus

$$(\vee_{i \in I} \tilde{\lambda}_i)(x) = \vee_{i \in I} (\tilde{\lambda}_i(x))$$

$$\geq \vee_{i \in I} (\tilde{\lambda}_i(x * y) \wedge \tilde{\lambda}_i(y) \wedge [0.5, 0.5])$$

$$= (\vee_{i \in I} \tilde{\lambda}_i)(x * y) \wedge (\vee_{i \in I} \tilde{\lambda}_i)(y) \wedge [0.5, 0.5]$$

Therefore

$$(\vee_{i \in I} \tilde{\lambda}_i)(x) \geq (\vee_{i \in I} \tilde{\lambda}_i)(x * y) \wedge (\vee_{i \in I} \tilde{\lambda}_i)(y) \wedge [0.5, 0.5].$$

Hence, $\vee_{i \in I} \tilde{\lambda}_i$ is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.

6. CONCLUSION

To investigate the structure of an algebraic system, we see that the interval valued fuzzy ideals with special properties always play a fundamental role.

In this paper, we introduce the concept of (α, β)-interval valued fuzzy ideals in BF-algebra, where α, β are any one of $\in$, q, $\in \vee q$, $\in \wedge q$ and investigate some of their related properties. We prove that every $(\in \vee q, \in \vee q)$-interval valued fuzzy ideal of a BF-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X. We show that when an $(\in, \in \vee q)$-interval valued fuzzy ideal of a BF-algebra X is an $(\in, \in)$-interval valued fuzzy ideal of X. We also prove that the intersection and union of any family of $(\in, \in \vee q)$-interval valued fuzzy ideals of a BF-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy ideal of X.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of interval valued fuzzy BF-algebras and their applications in other branches of algebra. In the future study of interval valued fuzzy BF-algebras, perhaps the following topics are worth to be considered:

(1) To characterize other classes of BF-algebras by using this notion;
(2) To apply this notion to some other algebraic structures;
(3) To consider these results to some possible applications in computer sciences and information systems in the future.

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Muhammad Idrees
Email: idrees.math@hotmail.com

Abdul Rehman
Email: abdul_maths@yahoo.com

Muhammad Zulfiqar
Email: mzulfiqarshafi@hotmail.com

Sardar Muhammad
Email: sardar_77@yahoo.com

Department of Mathematics[1]
University of Balochistan Quetta
Pakistan

&

Department of Mathematics[2]
GC University Lahore
Pakistan